MATH6031 Lecture 6

§ What's mirror symmetry?

origin : string theory ; discovered around 1987-1989.

spacetime $= \mathbb{R}^{3,1} \times X^6$

where X is a Calabi-Yau manifold (i.e. X admits a Kähler Ricci-flat metric and has holonomy $SU(3)$).

$$X \longmapsto \begin{cases} \text{Type IIA string theory } S_{IIA}(X) \\ \text{Type IIB string theory } S_{IIB}(X) \end{cases}$$

(both are examples of superconformal field theories (SCFTs)).

Physical def^{n} of mirror symmetry

Two Calabi-Yau manifolds X and $\check{X}$ are mirror to each other if

$$S_{IIA}(X) \overset{\text{dual}}{\simeq} S_{IIB}(\check{X})$$

interchanging A- & B-models

Witten : To a string theory, one can associate two topological conformal field theories (TCFTs)

$$S(X) \begin{cases} \rightsquigarrow A(X) & \text{A-model (symplectic geometry on } X) \\ \rightsquigarrow B(X) & \text{B-model (complex geometry on } X) \end{cases}$$

So, in mathematical terms, mirror symmetry predicts that

$$X \overset{\text{mirror}}{\longleftrightarrow} \check{X}$$

$$\Rightarrow \begin{cases} \underset{\text{sympl. geom.}}{A(X)} \simeq \underset{\text{cpx. geom.}}{B(\check{X})} \\ \underset{\text{cpx. geom.}}{B(X)} \simeq \underset{\text{sympl. geom.}}{A(\check{X})} \end{cases}$$

What does this mean?

First of all, this has the following implication for

Hodge numbers :

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(X) & & 0 & \\ 1 & & h^{2,1}(X) & & h^{1,2}(X) & & 1 \\ & 0 & & h^{2,2}(X) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \rightsquigarrow \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(\check{X}) & & 0 & \\ 1 & & h^{2,1}(\check{X}) & & h^{1,2}(\check{X}) & & 1 \\ & 0 & & h^{2,2}(\check{X}) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$$\Longleftrightarrow \quad \begin{cases} h^{1,1}(X) = h^{2,1}(\check{X}) \\ h^{2,1}(X) = h^{1,1}(\check{X}) \end{cases}$$

§ Examples

Mirror symmetry has lead to surprising predictions in enumerative geometry.

① Quintic 3-fold

$$X = \{ f = 0 \} \subset \mathbb{P}^4 \ , \quad f \in \mathbb{C}[x_0, x_1, \dots, x_4] \text{ homog.}$$
$$\deg f = 5 .$$

Candelas et al (1991) : mirror symmetry can be used to compute the numbers

$$n_d := \begin{array}{l} \# \text{ of rational curves of} \\ \deg = d \in H_2(X;\mathbb{Z}) \cong \mathbb{Z} \\ \text{in } X \end{array}$$

Known results up to 1991 :

$n_1 = 2{,}875$ (Schubert 1879)

$n_2 = 609{,}250$ (Katz 1986)

$n_3 = \cancel{2{,}682{,}549{,}425}$ (Ellingsrud & Strømme 1991)

$317{,}206{,}375$

More precisely, we consider

$$(\mathbb{Z}/5\mathbb{Z})^5 \curvearrowright \mathbb{P}^4 \quad \text{diagonally}$$

$$(x_0,\dots,x_4)\mapsto(\xi^{a_0}x_0,\dots,\xi^{a_4}x_4),\ \xi=e^{\frac{2\pi i}{5}}$$

and $(\mathbb{Z}/5\mathbb{Z})=\{(a,\dots,a)\mid a\in\mathbb{Z}\}$ acts trivially

$$\Rightarrow\quad (\mathbb{Z}/5\mathbb{Z})^5/(\mathbb{Z}/5\mathbb{Z})\curvearrowright\mathbb{P}^4$$

Now consider

$$X_\psi=\{x_0^5+\dots+x_4^5-5\psi x_0x_1\cdots x_4=0\}\subset\mathbb{P}^4$$

Then the subgroup $G=\{(a_0,\dots,a_4)\mid\sum_i a_i=0\}$ $\cong(\mathbb{Z}/5\mathbb{Z})^3$ acts on X_ψ.

The quotient X_ψ/G has 125 singular pts.

The mirror quintic is given by a crepant resolution

$$\check{X}=\check{X}_\psi:=\widetilde{X_\psi/G}$$

- To compute n_d's, we need to study the deform^{n} theory of complex structures on $\check{X}_\psi=\check{X}_z$

More precisely, we need to solve the Picard-Fuchs equation:

$$\left[\Theta^4-5z(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4)\right]\Phi=0$$

where $\Theta=z\frac{d}{dz}$ and $z=(5\psi)^{-5}$.

a kind of hypergeometric equations.

advantage: have explicit solutions

Let $\{\Phi_0(z),\Phi_1(z),\Phi_2(z),\Phi_3(z)\}$ be a basis of sol^{ns}

$$\Phi_0(z)=\sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}z^n,\qquad \Phi_1(z)=\Phi_0(z)\log z+\Pi(z)$$

hdom. fcn in z

- Define the mirror map as

$$q=f(z)=\exp\left(\frac{\Phi_1(z)}{\Phi_0(z)}\right)=\exp\left(\log z+\frac{\Pi(z)}{\Phi_0(z)}\right)=z(1+\cdots)$$

(a change of coordinates)

- Then we have the following conjecture by Candelas et al:

$$5 + \sum_{d=1}^{\infty} d^3 n_d \frac{q^d}{1-q^d} \underset{\text{via } q = f(z)}{=\!=} \int_{\check{X}_z} \Omega(z) \wedge \left(\frac{d}{dz}\right)^3 \Omega(z)$$

Here, $\Omega(z)$ is a holom. $(3,0)$-form on $\check{X}_z$

$\frac{d}{dz}\Omega(z)$: $(2,1)$-form

$\left(\frac{d}{dz}\right)^2 \Omega(z)$: $(1,2)$-form

$\left(\frac{d}{dz}\right)^3 \Omega(z)$: $(0,3)$-form

Refs : • Mirror symmetry and algebraic geometry by Cox and Katz

• Calabi-Yau manifolds and related geometries (Chapter 2 by Mark Gross).

② Local $\mathbb{P}^2$ (noncompact Calabi-Yau 3-fold)

$$X = \text{total space of } K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$$

This is called the local $\mathbb{P}^2$ because if $\mathbb{P}^2 \subset Y$ where Y is a cpt CY 3-fold then $\underset{\substack{\parallel \\ \text{tubular nbh} \\ \text{of } \mathbb{P}^2 \text{ in } Y}}{N_{\mathbb{P}^2/Y}} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ by adjunction

$$\check{X} = \check{X}_t = \left\{ uv = 1 + z_1 + z_2 + \frac{t}{z_1 z_2} \right\} \subset \mathbb{C}^2_{u,v} \times (\mathbb{C}^\times)^2_{z_1, z_2}$$

This mirror symmetry can be used to compute certain no. of rational curves, or more precisely, the local Gromov-Witten (GW) invariants :

Witten (GW) invariants :

$$N_{g,d}(X) := \int_{[\overline{\mathcal{M}}_{g,0}(K_{\mathbb{P}^2}, d)]^{vir}} \mathbb{1}$$

- To do so, we need to solve the Picard-Fuchs eqn

$$\left[\Theta^3 + 3t\Theta(3\Theta+1)(3\Theta+2)\right]\Phi = 0$$

$$\Phi = \int_{\Gamma_t} \Omega_{\check{X}_t} \text{ — period integrals}$$

$$[\Gamma_t] \in H_3(\check{X}_t)$$

where $\Theta = t\frac{d}{dt}$, $t \in \mathcal{M}_{\mathbb{C}}(\check{X})$

$\sim$ a hypergeometric eqn

- A basis of sol^{ns} is given by

$$\Phi_0 = 1, \quad \Phi_1 = \log t + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{(3k)!}{k!^3} t^k$$

$$\Phi_2 = (\log t)^2 + \cdots$$

- The mirror map is the change of coordinates

$$q = f(t) = \exp\left(\frac{\Phi_1(t)}{\cancel{\Phi_0(t)}}\right) = t(1+\cdots) : \mathcal{M}_{\mathbb{C}}(\check{X}) \to \mathcal{M}_K(X)$$

- Then mirror symmetry predicts that

$$(\log q)^2 + 3q\frac{d}{dq}\Big(\underbrace{\sum_{d=1}^{\infty} N_{0,d}(X)\, q^d}_{= \, d \cdot n_d \cdot \frac{q^d}{1-q^d}}\Big) \underset{q}{=\!=} \Phi_2(t)$$

via $q = f(t)$

③ <u>Non-CY setting</u>

$X = \mathbb{P}^2$, then mirror is <u>NOT</u> a mfd !

rather, its given by a so-called **Landau-Ginzburg model** $(\check{X}, W)$

where $\check{X} = (\mathbb{C}^\times)^2$

Landau-Ginzburg model $(\check{X}, W)$

where $\begin{cases} \check{X} = (\mathbb{C}^\times)^2 \\ W = z_1 + z_2 + \frac{q}{z_1 z_2} \end{cases}$

We call W the superpotential of the LG model

i.e. $X = \mathbb{P}^2 \xleftrightarrow{\text{mirror}} (\check{X}, W)$

Mirror symmetry predicts that

sympl. geom. on $\mathbb{P}^2$ $\simeq$ cpx geom. on $(\check{X}, W)$

For instance, we have the isomorphism of algebras

Prop $QH^*(\mathbb{P}^2) \simeq \mathrm{Jac}(W) := \mathbb{C}[\check{X}]/\mathcal{J}_W$

quantum cohomology of $\mathbb{P}^2$ (a deform of $H^*(\mathbb{P}^2)$)

Jacobian ring of W

Jacobian ideal of W

$\mathbb{C}[\check{X}]/\mathcal{J}_W = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}] \Big/ \left\langle z_1 \frac{\partial W}{\partial z_1}, z_2 \frac{\partial W}{\partial z_2} \right\rangle$

Pf: $H^*(\mathbb{P}^2) \cong \mathbb{C}[H]/\langle H^3 \rangle$

H^3 = $H \cup H \cup H$

and $H * H * H = q$

$*$ = quantum product

$H * H * H = q$ $\because \exists !$ line passing through 2 pts in $\mathbb{P}^2$

$\Rightarrow QH^*(\mathbb{P}^2) \cong \mathbb{C}[H]/\langle H^3 - q \rangle$

On the other hand,

$$\mathrm{Jac}(W) = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}] \Big/ \left\langle z_1 - \frac{q}{z_1 z_2}, z_2 - \frac{q}{z_1 z_2} \right\rangle$$

$$\cong \mathbb{C}[z]/\langle z^3 - q \rangle. \quad \#$$

More generally, we have

Thm (Givental, Lian-Liu-Yau, Barannikov-Kontsevich, ...)

(genus 0

Thm (Givental, Lian-Liu-Yau, Barannikov-Kontsevich, ...)

$$\mathrm{Frob}_A(X) \simeq \mathrm{Frob}_B(\check{X}, W)$$ (genus 0 mirror symmetry)

Big Question: WHY mirror symmetry works?